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Thirring sine-Gordon relationship by canonical methods

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Abstract. Using the canonical method developed for anomalous theories, we present the independent rederivation of the quantum relationship between the massive Thirring and the sine-Gordon models. The same method offers the possibility to obtain the Mandelstam soliton operators as a solution of the Poisson brackets "equation" for the fermionic fields. We checked the anticommutation and basic Poisson brackets relations for these composite operators. The transition from the Hamiltonian to the corresponding Lagrangian variables produces the known Mandelstam's result.

1 Introduction

The connection between the massive Thirring model of interacting fermions and the sine-Gordon model with non-linear scalar field is well known [1–3]. This Bose–Fermi equivalence has been obtained in [1] by performing the computations of the Green's functions for both theories. After identification of some parameters, the Green's functions became equal to a perturbation series, so that under these conditions these two theories are identical. An important step towards obtaining this result has been achieved in a pioneering paper [4]. In [2] Mandelstam has constructed the Fermi fields as non-local functions of the sine-Gordon scalars. He showed that the corresponding operators create and annihilate the bare sine-Gordon solitons. These operators satisfy the proper commutation relations as well as the Thirring model field equations, which confirms Coleman's result. This equivalence has been established on the quantum level and the relation between the Fermi and Bose fields is non-local. Beside the approaches mentioned above, Fermi–Bose equivalence was obtained in [5,6] using the quantum mechanical interaction picture and the Krein realization of the massless scalar field. The same problem has been considered in [7].

In this paper we are going to derive the above connection between massive Thirring and sine-Gordon models using the canonical method [8–10]. Starting with the fermionic Thirring model we are going to construct the equivalent bosonic theory, which appears to be the sine-Gordon one. Our approach is different from the previously mentioned ones and naturally works in the *Hamiltonian* formalism. It gives a simpler proof of the same result.

We consider the formulation of the Thirring model with auxiliary vector fields, which in virtue of the equations of motion gives the standard form of the Thirring model action. It is more convenient, regarding the fact that the method which we use is based on the canonical formalism. The form of the action with auxiliary fields becomes linear in the fermionic current j_μ. In Sect. 2.1 we are going to canonicaly quantize the fermionic field. So it is useful to keep all parts which contain this field and to omit the bilinear part in auxiliary field. Such a Lagrangian is invariant under local abelian gauge transformations. Consequently, the first class constraints (FCC) $j_\pm$ are present in the theory and satisfy the abelian algebra as a Poisson bracket (PB) algebra. The quantum theory is anomalous, so that the *central term* appears in the commutator algebra of the operators $\hat{j}_\pm$, and the constraints become second class (SCC).

We define the effective bosonized theory as a classical theory whose PB algebra of the constraints $J_\pm$ is isomorphic to the commutator algebra of the operators $\hat{j}_\pm$ in the quantized fermionic theory. Also, the bosonic Hamiltonian depends on $J_\pm$ in the same way as the fermionic Hamiltonian depends on $j_\pm$. The bosonized theory incorporates anomalies of the quantum fermionic theory at the classical level.

In Sect. 2.3 we find the effective Lagrangian for the given algebra as its PB current algebra and given the Hamiltonian in terms of the currents. Similar problems have been solved before in the literature [10] using canonical methods. We introduce the phase space coordinates φ, π and parameterize the constraints $J_\pm$ by them. Then we find the expressions for the constraints $J_\pm$ in terms of the phase space coordinates, satisfying the given PB algebra, as well as for the Hamiltonian density $\mathcal{H}_c$. We then use the general canonical method [8–10] for constructing the effective Lagrangian with the known representation of the constraints. Eliminating the momentum variable on invoking its equation of motion we obtain the Bose theory which is equivalent to the quantum Fermi theory. Finally, returning the omitted term, bilinear in A_μ, and eliminating the auxiliary vector field on invoking its equation of motion we obtain the sine-Gordon model.

By the way, we obtain Hamiltonian bosonization formulae for the currents which depend on the momenta, while those for scalar densities depend only on the coordinates. Known Lagrangian bosonization rules can be obtained from the Hamiltonian ones, after eliminating the momenta.

The massless Thirring model is considered separately in Sect. 2.4. It is shown that in its quantum action there exists one parameter which does not appear in the classical one. Therefore, the quantum massless Thirring model is non-uniquely defined in agreement with [12].

In Sect. 3. the same method will be applied for the construction of the fermionic Mandelstam's operators. The algebra of the currents is the basic PB algebra. Commutation relations between the currents and fermionic fields completely define the fermionic fields. So, we first find the PB between $j_\pm$ and $\psi_\pm$. The corresponding commutation relations of the operators $\hat{j}_\pm$ and $\hat{\psi}_\pm$ are not anomalous. In order to obtain the bosonized expression for the fermions we "solved" the PB equation, which is isomorphic to the previous operator relation. We find the representation for the unknown fermionic field using the known representation for the currents $J_\pm$. The solution depends on the phase space coordinates φ and π and represents the Hamiltonian form of Mandelstam's creation and annihilation operators.

Section 4 is devoted to concluding remarks. The derivation of the central term, using the normal ordering prescription, is presented in Appendix A, and the field product regularization in Appendix B.

2 Thirring model

In this section the canonical method of bosonization will be applied to the Thirring model.

2.1 Canonical analysis of the theory

The Thirring model [11] is a theory of the massive Dirac field in two-dimensional space-time defined by the following Lagrangian:

$$\mathcal{L}_{\rm Th} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi - \frac{g}{2} j_\mu j^\mu \,, \tag{2.1}$$

where g is coupling constant, and $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ is the fermionic current. In two-dimensional space-time the γ matrices are defined in terms of the Pauli matrices σ_1, σ_2 and σ_3 by $\gamma^0 = \sigma_1$, $\gamma^1 = -\mathrm{i}\sigma_2$, $\gamma_5 = -\mathrm{i}\sigma_1\sigma_2 = \sigma_3$ and obey the standard relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \,, \qquad \gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0 \,. \tag{2.2}$$

The metric tensor $\eta^{\mu\nu}$ is defined by $\eta^{00} = -\eta^{11} = 1$; $\eta^{01} = \eta^{10} = 0$. The axial-vector product $\gamma^\mu\gamma_5$ can be expressed in terms of γ^ν in the following way:

$$\gamma^\mu\gamma_5 = -\epsilon^{\mu\nu}\gamma_\nu, \tag{2.3}$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor $\epsilon^{01} = -\epsilon^{10} = 1$. The Weyl or chiral spinors are defined using the γ_5 matrix:

$$\gamma_5\psi_\pm = \mp\psi_\pm, \tag{2.4}$$

which can be expressed with the help of the chiral projectors $P_\pm \equiv \frac{1\mp\gamma_5}{2}$ as

$$P_\pm\psi_\pm = \pm\psi_\pm. \tag{2.5}$$

The definition of the projectors $P_\pm$ implies that the Dirac spinor ψ expressed in terms of the Weyl spinors $\psi_\pm$ has the form

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \tag{2.6}$$

The Lagrangian given by (2.1) is on-shell equivalent to the following one:

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}j^\mu A_\mu + \frac{1}{8g}A^\mu A_\mu \,. \tag{2.7}$$

Namely, the equations of motion for the auxiliary field A_μ which are obtained from the Lagrangian (2.7) have the form

$$\frac{1}{2}j^\mu + \frac{1}{4g}A^\mu = 0, \tag{2.8}$$

which, after substitution in (2.7), gives the Lagrangian (2.1).

We are going to quantize the fermionic field, so we will consider the Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}j^\mu A_\mu \,, \tag{2.9}$$

keeping the terms with fermionic fields. The canonical method of bosonization will be applied to this Lagrangian. In terms of the Weyl spinors $\psi_\pm$ and the light-cone components of the auxiliary field A_μ it reads

$$\begin{aligned}\mathcal{L}_0 = {}& \mathrm{i}\psi_-^*\dot{\psi}_- + \mathrm{i}\psi_+^*\dot{\psi}_+ + \mathrm{i}\psi_-^*\psi_-' - \mathrm{i}\psi_+^*\psi_+' \\ & -m(\psi_-^*\psi_+ + \psi_+^*\psi_-) + \frac{1}{2}(j_+A_- + j_-A_+) \,, \end{aligned} \tag{2.10}$$

where the chiral currents $j_\pm$ are defined by $j_\pm \equiv \sqrt{2}\psi_\pm^*\psi_\pm$, and the fields $A_\pm \equiv (1/\sqrt{2})(A_0 \pm A_1)$. The time and space coordinates are respectively $\tau \equiv x^0$ and $\sigma \equiv x^1$, and the corresponding derivatives are $\dot{\psi} \equiv \frac{\partial\psi}{\partial\tau}$ and $\psi' \equiv \frac{\partial\psi}{\partial\sigma}$. Now we will investigate the Hamiltonian structure of the theory defined by the Lagrangian (2.10). This Lagrangian is already in Hamiltonian form. It is linear in the time derivatives of the basic Lagrangian variables ψ_+ and ψ_-, whose conjugate momenta are $\pi_\pm = \mathrm{i}\psi_\pm^*$. Variables without time derivatives, A_+ and A_-, are Lagrange multipliers and the primary constraints corresponding to them are the FCC:

$$j_\pm \equiv \sqrt{2}\psi_\pm^*\psi_\pm = -\mathrm{i}\sqrt{2}\pi_\pm\psi_\pm \,. \tag{2.11}$$

From (2.10) we can conclude that the canonical Hamiltonian density of the Thirring model takes the form

$$\begin{aligned}\mathcal{H}_c &= -\mathrm{i}(\psi_-^*\psi_-' - \psi_+^*\psi_+') + m(\psi_-^*\psi_+ + \psi_+^*\psi_-) \\ &= t_+ - t_- + m(\rho_+ + \rho_-)\,,\end{aligned} \tag{2.12}$$

where we introduced the energy-momentum tensor $t_\pm$ and the chiral densities $\rho_\pm$ by the relations

$$t_\pm \equiv \mathrm{i}\psi_\pm^*\psi_\pm' = \pi_\pm\psi_\pm'\,, \quad \rho_\pm \equiv \psi_\pm^*\psi_\mp = -\mathrm{i}\pi_\pm\psi_\mp\,. \tag{2.13}$$

The total Hamiltonian is defined by

$$H_\mathrm{T} = \int \mathrm{d}\sigma \mathcal{H}_\mathrm{T}, \tag{2.14}$$

where the total Hamiltonian density, $\mathcal{H}_\mathrm{T}$, is

$$\mathcal{H}_\mathrm{T} = t_+ - t_- + m(\rho_+ + \rho_-) - \frac{1}{2}(j_+A_- + j_-A_+)\,. \tag{2.15}$$

Starting with the basic PB

$$\{\psi_\pm(\sigma), \pi_\pm(\bar\sigma)\} = \delta(\sigma - \bar\sigma)\,, \tag{2.16}$$

it is easy to show that the currents $j_\pm$ satisfy the two independent abelian PB algebras

$$\{j_\pm(\sigma), j_\pm(\bar\sigma)\} = 0\,, \quad \{j_+(\sigma), j_-(\bar\sigma)\} = 0\,. \tag{2.17}$$

Using (2.16), we can find the PB of the currents $j_\pm$ with the quantities $t_\pm$ and $\rho_\pm$:

$$\begin{aligned}&\{j_\pm(\sigma), t_\pm(\bar\sigma)\} = -j_\pm(\sigma)\delta'(\sigma-\bar\sigma) \\ &\{j_\pm(\sigma), t_\mp(\bar\sigma)\} = 0\,,\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\{j_\pm(\sigma), \rho_\pm(\bar\sigma)\} = -\mathrm{i}\sqrt{2}\rho_\pm\delta(\sigma-\bar\sigma), \\ &\{j_\pm(\sigma), \rho_\mp(\bar\sigma)\} = \mathrm{i}\sqrt{2}\rho_\mp\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{2.19}$$

The last relations imply

$$\{\mathcal{H}_c(\sigma), j_\pm(\bar\sigma)\} = \pm j_\pm(\sigma)\delta'(\sigma-\bar\sigma) \mp \mathrm{i}m\sqrt{2}(\rho_+ - \rho_-)\,, \tag{2.20}$$

which help us to obtain

$$\dot{j}_+ = \{j_+, H_\mathrm{T}\} = j_+' + \mathrm{i}m\sqrt{2}(\rho_+ - \rho-)\,, \tag{2.21}$$

$$\dot{j}_- = \{j_-, H_\mathrm{T}\} = -j_-' - \mathrm{i}m\sqrt{2}(\rho_+ - \rho-)\,. \tag{2.22}$$

Taking the sum of (2.21) and (2.22), we get

$$\partial_- j_+ + \partial_+ j_- = \partial_\mu j^\mu = 0\,, \tag{2.23}$$

which implies that the current j^μ is conserved.

Since the constraints $j_\pm$ are first class ones, the classical theory, defined by the Lagrangian (2.9), has a local abelian symmetry, whose generators $j_\pm$ satisfy the abelian PB algebra given by (2.17).

2.2 Quantization of the fermionic theory

The passage from the classical to the quantum theory can be obtained by introducing the operators $\hat\psi$ and $\hat\pi$, instead of the corresponding classical fields. The Poisson brackets are replaced by the corresponding commutators, and the operator product is defined using the normal ordering prescription, whose details are explained in Appendix A. We have

$$\hat{j}_\pm \equiv -\mathrm{i}\sqrt{2} : \hat\pi_\pm\hat\psi_\pm :, \quad \hat{t}_\pm \equiv : \hat\pi_\pm\hat\psi_\pm' :, \quad \hat\rho \equiv -\mathrm{i} : \hat\pi_\pm\hat\psi_\mp : \,. \tag{2.24}$$

The algebra of the operators $\hat{j}_\pm$, $\hat{t}_\pm$ and $\hat\rho_\pm$ takes the form (Appendix A)

$$\begin{aligned}&\left[\hat{j}_\pm(\sigma), \hat{j}_\pm(\bar\sigma)\right] = \pm 2\mathrm{i}\hbar\kappa\,\delta'(\sigma-\bar\sigma), \\ &\left[\hat{j}_+(\sigma), \hat{j}_-(\bar\sigma)\right] = 0\,,\end{aligned} \tag{2.25}$$

$$\begin{aligned}&\left[\hat{j}_\pm(\sigma), \hat{t}_\pm(\bar\sigma)\right] = -\mathrm{i}\hbar\hat{j}_\pm(\bar\sigma)\delta'(\sigma-\bar\sigma), \\ &\left[\hat{j}_\pm(\sigma), \hat{t}_\mp(\bar\sigma)\right] = 0\,,\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\left[\hat{j}_\pm(\sigma), \hat\rho_\pm(\bar\sigma)\right] = \hbar\sqrt{2}\hat\rho_\pm\delta(\sigma-\bar\sigma), \\ &\left[\hat{j}_\pm(\sigma), \hat\rho_\mp(\bar\sigma)\right] = -\hbar\sqrt{2}\hat\rho_\mp\delta(\sigma-\bar\sigma)\,,\end{aligned} \tag{2.27}$$

where $\kappa \equiv \frac{\hbar}{2\pi}$.

The current operators with different chirality commute, as well as the corresponding variables in the classical theory. The difference between the classical and the quantum algebra is the appearance of the central term in the commutator current algebra (2.25). As a consequence, the operators $\hat{j}_\pm$ are second class constraints operators. This leads to the existence of the anomaly. Namely, the symmetry of the classical theory, whose generators are first class constraints $j_\pm$, is no longer a symmetry at the quantum level.

2.3 Effective bosonic theory

Now we will introduce new variables $J_\pm, \Theta_\pm, R_\pm$ and postulate their PB algebra to be isomorphic to the commutator algebra in the quantum fermionic theory, given by (2.25), (2.26) and (2.27):

$$\begin{aligned}&\{J_\pm(\sigma), J_\pm(\bar\sigma)\} = \pm 2\kappa\delta'(\sigma-\bar\sigma), \\ &\{J_+(\sigma), J_-(\bar\sigma)\} = 0\,,\end{aligned} \tag{2.28}$$

$$\begin{aligned}&\{\Theta_\pm(\sigma), J_\pm(\bar\sigma)\} = J_\pm(\sigma)\,\delta'(\sigma-\bar\sigma), \\ &\{\Theta_\pm(\sigma), J_\mp(\bar\sigma)\} = 0\,,\end{aligned} \tag{2.29}$$

$$\begin{aligned}&\{J_\pm(\sigma), R_\pm(\bar\sigma)\} = -\mathrm{i}\sqrt{2}R_\pm(\sigma)\delta(\sigma-\bar\sigma), \\ &\{J_\pm(\sigma), R_\mp(\bar\sigma)\} = \mathrm{i}\sqrt{2}R_\mp(\sigma)\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{2.30}$$

Let us find the expressions for the currents $J_\pm$, the energy-momentum tensor $\Theta_\pm$ and the chiral densities $R_\pm$ in terms of the scalar field φ and its conjugate momenta π, with the PB

$$\{\varphi(\sigma), \pi(\bar\sigma)\} = \delta(\sigma-\bar\sigma). \tag{2.31}$$

Assuming that the currents $J_\pm$ are linear in the momentum π, it is easy to show that the expression

$$J_\pm = \pm\pi + \kappa\varphi' \tag{2.32}$$

is a solution of (2.28). Supposing that the energy-momentum tensor $\Theta_\pm$ is quadratic in the currents $J_\pm$, we can immediately obtain its bosonic representation from the algebra (2.29):

$$\Theta_\pm = \pm \frac{1}{4\kappa} J_\pm J_\pm \,. \tag{2.33}$$

The bosonic representation for scalar densities, $R_\pm$, can be obtained from the algebra (2.30). Assuming that the scalar densities are momentum independent, we have

$$R_\pm = M \exp(\pm \mathrm{i}\sqrt{2}\varphi)\,, \tag{2.34}$$

where M is constant. The scalar densities as well as the parameter M have the dimension of mass.

The total Hamiltonian density of the effective bosonic theory is defined by the analogy with the total Hamiltonian density of the fermionic theory, given by (2.15):

$$\mathcal{H}_\mathrm{T} = \Theta_+ - \Theta_- + m(R_+ + R_-) - \frac{1}{2}(J_+ A_- + J_- A_+)\,, \tag{2.35}$$

and the Lagrangian of the effective bosonic theory has the form

$$\mathcal{L}_0^\mathrm{Th} = \pi\dot{\varphi} - \mathcal{H}_\mathrm{T}\,. \tag{2.36}$$

Substituting (2.35) in (2.36), and using (2.32), (2.33), and (2.34), we obtain

$$\mathcal{L}_0^\mathrm{Th} = \pi\dot{\varphi} - \frac{1}{2\,\kappa}\pi^2 - \frac{1}{2}\,\kappa\,\varphi'^2 - 2mM\cos\sqrt{2}\varphi + \frac{1}{\sqrt{2}}(-\pi A_1 + \kappa\varphi' A_0)\,. \tag{2.37}$$

On invoking the equation of motion for the momentum π

$$\pi = \kappa\left(\dot{\varphi} - \frac{A_1}{\sqrt{2}}\right)\,, \tag{2.38}$$

this Lagrangian takes the form

$$\mathcal{L}_0^\mathrm{Th} = \frac{1}{2}\,\kappa\,\partial_\mu\varphi\partial^\mu\varphi + \frac{\kappa}{\sqrt{2}}\,\epsilon^{\mu\nu}A_\mu\partial_\nu\varphi - 2mM\cos(\sqrt{2}\,\varphi) + \frac{1}{4}\kappa A_1^2\,. \tag{2.39}$$

It is possible to add to the effective Lagrangian some local functional, depending on the fields A_+ and A_-. In order to obtain the Lorentz invariant action of the effective bosonic theory, we will choose an additional term in the form

$$\Delta\mathcal{L}_0^\mathrm{Th} = -\frac{1}{4}\,\kappa A_1^2. \tag{2.40}$$

Adding the counterterm $\Delta\mathcal{L}_0^\mathrm{Th}$ to the Lagrangian (2.39), and returning the term bilinear in A_μ we get

$$\mathcal{L}_\mathrm{eff}^\mathrm{Th} = \frac{1}{2}\,\kappa\,\partial_\mu\varphi\partial^\mu\varphi + \frac{\kappa}{\sqrt{2}}\,\epsilon^{\mu\nu}A_\mu\partial_\nu\varphi - 2mM\cos(\sqrt{2}\,\varphi) + \frac{1}{8g}A^\mu A_\mu\,. \tag{2.41}$$

Now we can eliminate the auxiliary field A_μ from the Lagrangian (2.41), using its equation of motion

$$A^\mu = -\frac{4g\kappa}{\sqrt{2}}\,\epsilon^{\mu\nu}\partial_\nu\varphi\,. \tag{2.42}$$

Substituting this equation back into the Lagrangian (2.41), we get

$$\mathcal{L}_\mathrm{eff}^\mathrm{Th} = \frac{1}{2}\,\kappa(1+2g\kappa)\,\partial_\mu\varphi\partial^\mu\varphi - 2mM\cos(\sqrt{2}\,\varphi)\,. \tag{2.43}$$

This Lagrangian, after rescaling the scalar field φ, $\varphi \to \frac{\sqrt{2}}{\beta}\varphi$, takes the form

$$\mathcal{L}_\mathrm{eff}^\mathrm{Th} = \frac{1}{2}\,\partial_\mu\varphi\partial^\mu\varphi - 2mM\cos(\beta\,\varphi)\,, \tag{2.44}$$

where β is defined by

$$\beta \equiv \left[\frac{1}{2}\kappa(1+2g\kappa)\right]^{-1/2}\,. \tag{2.45}$$

As usual, we will add a constant term to the Lagrangian (2.44), in order to have vanishing energy for the vacuum configuration $\varphi = 0$ and obtain

$$\mathcal{L}_\mathrm{eff}^\mathrm{Th} = \frac{1}{2}\,\partial_\mu\varphi\partial^\mu\varphi - 2mM\,\left[\cos(\beta\,\varphi) - 1\right]\,. \tag{2.46}$$

This is the Lagrangian of the sine-Gordon theory. Therefore, the Thirring model is equivalent to the sine-Gordon theory, if there exists the following relation between the parameters:

$$\frac{4\pi}{\beta^2\,\hbar} = 1 + \frac{g\hbar}{\pi} = 1 + 2g\kappa\,, \tag{2.47}$$

which is consequence of (2.45). The result given by (2.47), obtained by the canonical method, is in agreement with the one in [1]. Coleman has obtained this result by direct computation of Green's functions in both the Thirring and the sine-Gordon model using a perturbative technique. The relation (2.47) is the main result in this chapter. From its form, we can easy conclude that the free Thirring model ($g = 0$) is equivalent to the sine-Gordon theory with

$$\beta^2 = \frac{4\pi}{\hbar}\,. \tag{2.48}$$

It is worth to emphasize that the relation (2.47) implies duality between the Thirring and sine-Gordon model. Namely, from this relation it follows that the large values of the Thirring coupling constant g corresponds to a small value of the sine-Gordon parameter β.

2.4 One parameter class solutions of the massless Thirring model

We shall consider separately the massless Thirring model. This case is specially interesting, because the corresponding quantum theory is non-uniquely defined. Namely, in the quantum action of the theory there exists one parameter, which does not appear in the classical one [12]. Since the Thirring sine-Gordon relationship was already established, we will show the existence of this parameter starting with the corresponding sine-Gordon model.

Firstly, we split the vector field from the Lagrangian (2.39) to the quantum and external part $A_\mu = a_\mu + A_\mu^{\text{ex}}$. Here the field a_μ plays the role of our auxiliary field and A_μ^{ex} is Hagen's external source. Then, omitting the mass term and the local functional dependence on the vector fields, we obtain from (2.39)

$$\mathcal{L}^{\text{Th}}(\varphi, a + A^{\text{ex}}) = \frac{1}{2}\kappa\,\partial_\mu\varphi\partial^\mu\varphi + \frac{\kappa}{\sqrt{2}}\epsilon^{\mu\nu}(a_\mu + A_\mu^{\text{ex}})\partial_\nu\varphi. \tag{2.49}$$

The invariance of the Thirring model under the replacement

$$j^\mu \to j_5^\mu\,, \qquad A_\mu^{\text{ex}} \to A_{5\mu}^{\text{ex}}\,, \qquad g \to -g\,, \tag{2.50}$$

corresponds to the Lagrangian $\mathcal{L}^{\text{Th}}(\varphi_5, a_5 + A_5^{\text{ex}})$. Note that we introduce new auxiliary fields φ_5 and a_5, while the external fields are related by the dual transformation $A_{5\mu}^{\text{ex}} = \varepsilon_{\mu\nu}A^{\text{ex}\nu}$.

The symmetry of the massless Thirring model given by (2.50) allows us to introduce the one-parameter Lagrangian

$$\begin{aligned}\mathcal{L}(\xi,\eta) &= \xi\mathcal{L}^{\text{Th}}(\varphi, a + A^{\text{ex}}) + \eta\mathcal{L}^{\text{Th}}(\varphi_5, a_5 + A_5^{\text{ex}}) \\ &\quad + \frac{1}{8g}(a_\mu a^\mu - a_{5\mu}a_5^\mu)\,,\end{aligned} \tag{2.51}$$

with $\xi+\eta = 1$. The parameters ξ and η obey this constraint because the Lagrangian given by (2.51) has to correspond to the Lagrangian of the massless Thirring model. The terms quadratic in auxiliary fields in fact replaced the last term of (2.41). The second part was added with opposite sign according to the symmetry replacement $g \to -g$.

After elimination of all auxiliary fields a, a_5 and then φ, φ_5, we obtain the effective action

$$W(A^{\text{ex}}) = -\frac{\hbar}{8}\int \mathrm{d}^2x A_\mu^{\text{ex}} D_g^{\mu\nu} A_\nu^{\text{ex}}\,, \tag{2.52}$$

where

$$D_g^{\mu\nu} = \left(\varepsilon^{\mu\rho}\varepsilon^{\nu\sigma}\frac{\xi}{1+\frac{g\xi\hbar}{\pi}} + \eta^{\mu\rho}\eta^{\nu\sigma}\frac{\eta}{1-\frac{g\eta\hbar}{\pi}}\right)\partial_\rho\partial_\sigma\frac{1}{\partial^2}\,. \tag{2.53}$$

Up to the normalization factor this is just relation (3.13) from Hagen's paper [12].

The expression for the effective action is equivalent to the solution of the functional integral [10],

$$\langle 0 \mid 0\rangle_{A,g} = \int \mathrm{d}\bar{\psi}\mathrm{d}\psi \mathrm{e}^{\mathrm{i}\mathcal{L}_{\text{Th}}} = \mathrm{e}^{\mathrm{i}W(A^{\text{ex}})}\,, \tag{2.54}$$

which corresponds to (3.12) in the first of [12]. Using these expressions it is easy to reproduce the other Hagen results.

3 Bosonization of fermionic fields

In this section we will apply the canonical method of bosonization to the fermionic fields. Starting with the PB of the fermionic fields $\psi_\pm$ and the corresponding momenta $\pi_\pm$ with currents $j_\pm$, the fermionic fields will be expressed in terms of the bosonic phase space coordinates φ and π. After quantization, these classical fermionic fields become the operators. In order to show that these operators are really fermionic ones, we investigate their anticommutation relations. From the bosonic form of the operators $\Psi_\pm$, we easily obtain the bosonic representation of the scalar and pseudoscalar density $\hat{\bar{\Psi}}\hat{\Psi}$ and $\hat{\bar{\Psi}}\gamma_5\hat{\Psi}$, respectively. These results are consistent with the ones obtained in [2].

3.1 Construction of the fermionic field operators

The Poisson brackets of the fermionic fields $\psi_\pm$ and their conjugate momenta $\pi_\pm$ with the currents $j_\pm$ have the form

$$\begin{aligned}&\{j_\pm(\sigma), \psi_\pm(\bar\sigma)\} = \mathrm{i}\sqrt{2}\psi_\pm\delta(\sigma-\bar\sigma),\\ &\{j_\pm(\sigma), \psi_\mp(\bar\sigma)\} = 0\,,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\{j_\pm(\sigma), \pi_\pm(\bar\sigma)\} = -\mathrm{i}\sqrt{2}\pi_\pm\delta(\sigma-\bar\sigma),\\ &\{j_\pm(\sigma), \pi_\mp(\bar\sigma)\} = 0\,.\end{aligned} \tag{3.2}$$

Because the right hand side is linear in the fields, the anomaly is absent and, after quantization, the algebra of the operators $\hat\psi_\pm$, $\hat\pi_\pm$ and $\hat\jmath_\pm$ preserves the original form

$$\begin{aligned}&\left[\hat\jmath_\pm(\sigma), \hat\psi_\pm(\bar\sigma)\right] = -\hbar\sqrt{2}\hat\psi_\pm\delta(\sigma-\bar\sigma),\\ &\left[\hat\jmath_\pm(\sigma), \hat\psi_\mp(\bar\sigma)\right] = 0\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\left[\hat\jmath_\pm(\sigma), \hat\pi_\pm(\bar\sigma)\right] = \hbar\sqrt{2}\hat\pi_\pm\delta(\sigma-\bar\sigma),\\ &\left[\hat\jmath_\pm(\sigma), \hat\pi_\mp(\bar\sigma)\right] = 0\,.\end{aligned} \tag{3.4}$$

Now, we will construct the bosonic representation of the fermionic fields $\Psi_\pm$ and their conjugate momenta $\Pi_\pm$. We demand that the Poisson brackets algebra of the fields $\Psi_\pm$, their conjugate momenta $\Pi_\pm$, and currents $J_\pm$, whose bosonic form is already known, is isomorphic to the algebra of the operators $\hat\psi_\pm$, $\hat\pi_\pm$ and $\hat\jmath_\pm$, respectively. Therefore, we have

$$\begin{aligned}&\{J_\pm(\sigma), \Psi_\pm(\bar\sigma)\} = \mathrm{i}\sqrt{2}\Psi_\pm(\sigma)\delta(\sigma-\bar\sigma),\\ &\{J_\pm(\sigma), \Psi_\mp(\bar\sigma)\} = 0\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\{J_\pm(\sigma), \Pi_\pm(\bar\sigma)\} = -\mathrm{i}\sqrt{2}\Pi_\pm(\sigma)\delta(\sigma-\bar\sigma),\\ &\{J_\pm(\sigma), \Pi_\mp(\bar\sigma)\} = 0\,.\end{aligned} \tag{3.6}$$

In order to solve these equations in terms of $\Psi_\pm$ and $\Pi_\pm$, let us introduce the variables $I_\pm$ as follows:

$$I_\pm(\sigma) \equiv \int_{-\infty}^{\sigma}\mathrm{d}\sigma_1\, J_\pm(\sigma_1)\,. \tag{3.7}$$

With the help of the bosonic representation of the currents (2.32), we obtain the $I_\pm$ dependence of the basic bosonic variables

$$I_\pm(\sigma) = \pm \int_{-\infty}^{\sigma} \mathrm{d}\sigma_1\, \pi(\sigma_1) + \kappa\varphi(\sigma)\,. \tag{3.8}$$

Using the Poisson brackets current algebra, given by (2.28), we find the PB of the variables $I_\pm$ with the currents $J_\pm$,

$$\{J_\pm(\sigma), I_\pm(\bar\sigma)\} = \mp 2\kappa\, \delta(\sigma-\bar\sigma)\,, \tag{3.9}$$

$$\{J_\pm, I_\mp\} = 0\,. \tag{3.10}$$

Let us suppose that the fields $\Psi_\pm$ and their conjugate momenta depend only on the variables $I_\pm$. Under that assumption, from the algebra given by (3.5) and (3.6), we get the following equations:

$$\{J_\pm(\sigma), I_\pm(\bar\sigma)\}\, \frac{\partial \Psi_\pm}{\partial I_\pm} = \mathrm{i}\sqrt{2}\Psi_\pm(\sigma)\delta(\sigma-\bar\sigma)\,, \tag{3.11}$$

$$\{J_\pm(\sigma), I_\pm(\bar\sigma)\}\, \frac{\partial \Pi_\pm}{\partial I_\pm} = -\mathrm{i}\sqrt{2}\Pi_\pm(\sigma)\delta(\sigma-\bar\sigma). \tag{3.12}$$

With the help of (3.9), we obtain the bosonic representation of the fermionic field $\Psi_\pm$ and their conjugate momenta $\Pi_\pm$,

$$\Psi_\pm = C_\pm \exp\left(\mp \frac{\mathrm{i}}{\kappa\sqrt{2}}\, I_\pm\right), \tag{3.13}$$

$$\Pi_\pm = D_\pm \exp\left(\pm \frac{\mathrm{i}}{\kappa\sqrt{2}}\, I_\pm\right), \tag{3.14}$$

where $C_\pm$ and $D_\pm$ are the constants which will be determined using a regularization procedure. These constants are not independent. The relation $D_\pm = \mathrm{i}\, C^*_\pm$ follows from the classical fermionic theory constraints $\pi_\pm = \mathrm{i}\psi^*_\pm$ and their bosonic analogue $\Pi_\pm = \mathrm{i}\Psi^*_\pm$.

After quantization the classical fields $\Psi_\pm$ and their conjugate momenta $\Pi_\pm$ become operators $\hat\Psi_\pm$ and $\hat\Pi_\pm$. So, the normal ordering prescription has to be applied to the operators product in the right hand side in (3.13) and (3.14):

$$\hat\Psi_\pm = C_\pm : \exp\left(\mp \frac{\mathrm{i}}{\kappa\sqrt{2}}\, \hat I_\pm\right) :\,, \tag{3.15}$$

$$\hat\Pi_\pm = \mathrm{i}\, C^*_\pm : \exp\left(\pm \frac{\mathrm{i}}{\kappa\sqrt{2}}\, \hat I_\pm\right) : \,. \tag{3.16}$$

This means that after expansion of the exponent in the right hand side, annihilation operators are placed right to the creation ones (Appendix A).

As direct computation shows (Appendix B), products of the operators $\hat\Psi^*_\pm \hat\Psi_\pm$ at the same point of space are singular. In order to regularize these products, let us introduce the operators $\hat{\hat J}_\pm$, which are the products of field operators at the different points of space,

$$\hat{\hat J}_\pm(\sigma_1, \sigma_2) \equiv \sqrt{2}\hat\Psi^*_\pm(\sigma_1)\hat\Psi_\pm(\sigma_2)\,. \tag{3.17}$$

After some calculations (Appendix B), we obtain

$$\begin{aligned}\hat{\hat J}_\pm(\sigma, \sigma+\eta)|_{\eta\to 0} &\equiv \sqrt{2}\hat\Psi^*_\pm(\sigma)\hat\Psi_\pm(\sigma+\eta)|_{\eta\to 0} \\ &= \frac{F_\pm}{\eta \pm \mathrm{i}\varepsilon} + Z_\pm \hat J_\pm(\sigma) \quad (\varepsilon > 0)\,,\end{aligned} \tag{3.18}$$

where $F_\pm$ and $Z_\pm$ are given by following expressions (Λ is the cutoff parameter):

$$F_\pm = \pm \mathrm{i}\Lambda\sqrt{2}|C_\pm|^2, \qquad Z_\pm = \frac{\Lambda}{\kappa}|C_\pm|^2\,. \tag{3.19}$$

Because $\hat\Psi_\pm$ is a representation of the $\hat\psi_\pm$, we expect that a bilinear combination in (3.18) produces $\hat J_\pm$, since it is the same combination as in (2.11). So the natural choice for the constants is

$$Z_\pm = 1. \tag{3.20}$$

From the last relation, we get the values of the constants $F_\pm$ and $C_\pm$,

$$C_\pm = \sqrt{\frac{\kappa}{\Lambda}}\,, \qquad F_\pm = \pm \mathrm{i}\kappa\sqrt{2}\,. \tag{3.21}$$

With these values of the constants $F_\pm$ and $C_\pm$, the operators $\hat\Psi_\pm$, $\hat\Pi_\pm$ and $\hat{\hat J}_\pm$ take the form

$$\hat\Psi_\pm = \sqrt{\frac{\kappa}{\Lambda}} : \exp\left(\mp \frac{\mathrm{i}}{\kappa\sqrt{2}} \hat I_\pm\right) : \,, \tag{3.22}$$

$$\hat\Pi_\pm = \mathrm{i}\sqrt{\frac{\kappa}{\Lambda}} : \exp\left(\pm \frac{\mathrm{i}}{\kappa\sqrt{2}} \hat I_\pm\right) : \,, \tag{3.23}$$

$$\hat{\hat J}_\pm(\sigma, \sigma+\eta)|_{\eta\to 0} = \pm \frac{\mathrm{i}\kappa\sqrt{2}}{\eta \pm \mathrm{i}\varepsilon} + \hat J_\pm(\sigma)\,. \tag{3.24}$$

In order to compare these results with ones from [2], we will derive the Lagrangian form of the operators $\hat\Psi_\pm$. Note that up to this point in Sect. 3. we did not specify the Hamiltonian of the theory. So our canonical expression is valid for any theory satisfying PB algebra (3.1) and (3.2). Passing to the Lagrangian formulation we must be more specific, and we chose the example of the Thirring model, which we considered in the previous section. We will express the momentum π in terms of the corresponding velocity starting from the equation of motion for the momentum, (2.38), and using the equation of motion for the auxiliary field A_μ

$$\pi = \frac{2}{\beta^2}\dot\varphi\,. \tag{3.25}$$

Substituting the last equation in (3.13), after rescaling $\varphi \to \frac{\sqrt{2}}{\beta}\varphi$, we obtain the Lagrangian form of the fields $\Psi_\pm$,

$$\Psi_\pm(\sigma) = C_\pm \exp\left\{-\frac{\mathrm{i}}{\kappa\beta}\int_{-\infty}^{\sigma} \mathrm{d}\bar\sigma\dot\varphi(\bar\sigma) \mp \frac{\mathrm{i}\beta}{2}\varphi(\sigma)\right\}. \tag{3.26}$$

After quantization, from the last relation we get

$$\hat\Psi_\pm(\sigma) = C_\pm\ : \exp\left\{-\frac{\mathrm{i}}{\kappa\beta}\int_{-\infty}^{\sigma} \mathrm{d}\bar\sigma\hat{\dot\varphi}(\bar\sigma) \mp \frac{\mathrm{i}\beta}{2}\hat\varphi(\sigma)\right\} : \,. \tag{3.27}$$

This form of the field operators is in agreement with the one obtained in [2] from the requirement that the operators $\hat{\Psi}_\pm$ have to be annihilation operators for solitons in sine-Gordon theory, as well as that they anticommute with themselves. The result given by (3.22) is more general, because it is in Hamiltonian form, so it can be applied to the other two-dimensional models. Additionally, this result is obtained from a smaller number of assumptions. Namely, we obtained this result demanding that the fields $\Psi_\pm$ and the currents $J_\pm$ have to obey a Poisson brackets algebra which is isomorphic to the algebra of the operators $\hat{j}_\pm$ and $\hat{\psi}_\pm$.

3.2 Anticommutation relations for the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$

In this subsection we will show that the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$, given by (3.22) and (3.23), obey the canonical anticommutation relations.

In order to justify the interpretation of the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$ as the fermionic operators, we should show that they obey canonical anticommutation relations. Firstly, we will find anticommutation relations for the operators $\hat{\Psi}_\pm$. Using (3.22) and (B.3), we find

$$\hat{\Psi}_\pm(\sigma)\hat{\Psi}_\pm(\bar{\sigma}) = \frac{\kappa}{\Lambda}\exp\left\{-\frac{1}{2\kappa^2}\left[\hat{I}_\pm^{(\pm)}(\sigma), \hat{I}_\pm^{(\mp)}(\bar{\sigma})\right]\right\} \times :\exp\left\{\mp\frac{1}{2\kappa^2}\left[\hat{I}_\pm(\sigma) + \hat{I}_\pm(\bar{\sigma})\right]\right\}: (3.28).$$

With the help of (B.15), in the limit $\varepsilon \to 0$, we find that the anticommutator for the fields $\Psi_\pm$ vanishes:

$$\left[\hat{\Psi}_\pm(\sigma), \hat{\Psi}_\pm(\bar{\sigma})\right]_+ = 0\,. \tag{3.29}$$

The calculation, which is very similar to the previous one, shows that the anticommutator for the momenta also vanishes:

$$\left[\hat{\Pi}_\pm(\sigma), \hat{\Pi}_\pm(\bar{\sigma})\right]_+ = 0\,. \tag{3.30}$$

Now we will find the anticommutator for the fields $\Psi_\pm$ with their conjugate momenta $\Pi_\pm$. Using (3.22), (3.23) and (B.3), we obtain

$$\hat{\Psi}_\pm(\sigma)\hat{\Pi}_\pm(\bar{\sigma}) = \mathrm{i}\,\frac{\kappa}{\Lambda}\,\exp\left\{\frac{1}{2\kappa^2}\left[\hat{I}_\pm^{(\pm)}(\sigma), \hat{I}_\pm^{(\mp)}(\bar{\sigma})\right]\right\} \times :\exp\left\{\mp\frac{\mathrm{i}}{\kappa\sqrt{2}}\left[\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})\right]\right\}:,$$

and with the help of (B.14) we get

$$\hat{\Psi}_\pm(\sigma)\hat{\Pi}_\pm(\bar{\sigma}) = \mathrm{i}\hbar\,\delta^{(\pm)}(\sigma - \bar{\sigma}) \times :\exp\left\{\mp\frac{\mathrm{i}}{\kappa\sqrt{2}}\left[\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})\right]\right\}:,$$

which produces the canonical anticommutation relations for the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$:

$$\begin{aligned}&\left[\hat{\Psi}_\pm(\sigma), \hat{\Pi}_\pm(\bar{\sigma})\right]_+ \\ &= \mathrm{i}\hbar\delta(\sigma - \bar{\sigma}) :\exp\left\{\mp\frac{\mathrm{i}}{\kappa\sqrt{2}}\left[\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})\right]\right\}: \\ &= \mathrm{i}\hbar\delta(\sigma - \bar{\sigma})\,.\end{aligned} \tag{3.31}$$

3.3 Bosonization of the scalar densities

Products of the fermionic field operators $\hat{\bar{\Psi}}\hat{\Psi}$ and $\hat{\bar{\Psi}}\gamma^5\hat{\Psi}$ we will express in terms of the bosonic variables using (3.22). Note that the relation $\left[\hat{I}_+, \hat{I}_-\right] = 0$ simplifies the calculations. With the help of (B.3), we obtain

$$\hat{\Psi}_\pm^*\hat{\Psi}_\mp = \frac{\kappa}{\Lambda} :\exp\left[\frac{\pm\mathrm{i}}{\kappa\sqrt{2}}(\hat{I}_+ + \hat{I}_-)\right]: \,. \tag{3.32}$$

Substituting (3.8) in the last equation, we find

$$\hat{\Psi}_\pm^*\hat{\Psi}_\mp = \frac{\kappa}{\Lambda} :\exp\left(\pm\mathrm{i}\sqrt{2}\varphi\right): \,. \tag{3.33}$$

So, the bosonic representations for the scalar density $\hat{\bar{\Psi}}\hat{\Psi}$ and the pseudoscalar density $\hat{\bar{\Psi}}\gamma^5\hat{\Psi}$ are

$$\hat{\bar{\Psi}}\hat{\Psi} = \hat{\Psi}_-^*\hat{\Psi}_+ + \hat{\Psi}_+^*\hat{\Psi}_- = \frac{\hbar}{\pi\Lambda} :\cos\left(\sqrt{2}\varphi\right):, \tag{3.34}$$

$$\hat{\bar{\Psi}}\gamma^5\hat{\Psi} = -\hat{\Psi}_-^*\hat{\Psi}_+ + \hat{\Psi}_+^*\hat{\Psi}_- = \frac{\mathrm{i}\hbar}{\pi\Lambda} :\sin\left(\sqrt{2}\varphi\right): \,. \tag{3.35}$$

These results are consistent with the ones obtained by direct applying the method to the scalar densities.

4 Conclusion

In this paper we presented a complete and independent derivation of the Thirring sine-Gordon relationship, using the Hamiltonian methods. We also obtained the Hamiltonian and Lagrangian representation for the Mandelstam fermionic operators.

We started with a canonical analysis of the theory where fermions are coupled to the auxiliary external gauge field. The massive Thirring model can easily be obtained from this Lagrangian by adding the square of the auxiliary field and eliminating it on invoking its equation of motion. We found that there exists FCC $j_\pm$ in our theory, whose PB are equal to zero. In the quantum theory, the central term appears in the commutation relations of the operators $\hat{j}_\pm$. This changes the nature of the constraints because they become SCC.

We define the new effective theory, postulating the PB of the constraints and Hamiltonian density, following the method developed in [10]. We require that the classical PB algebra of the bosonic theory is isomorphic to the quantum commutator algebra of the fermionic theory. Then we found the representation for the currents and Hamiltonian density in terms of phase space coordinates. Finally, we derived the effective action using the general canonical formalism and obtained the equivalent bosonized model. Together with the auxiliary field term this is just the sine-Gordon action, up to the identifications of some parameters in agreement with [1,2]. For the massless Thirring model it is shown that its quantum effective action has one parameter which does not exist in the classical one. We determined the quantum action using formal invariance of the massless Thirring model under the replacements given by (2.50) and the already established Thirring sine-Gordon relationship.

The algebra of the currents $J_\pm$ is the basic PB algebra. Knowing the representation of the currents $J_\pm$ in terms of φ and π we can find the representation for all other quantities from their PB algebra with the currents. In Sect. 2. we found the bosonization rules for the chiral densities. The main result of Sect. 3. is the bosonic representation for the fermions, which has been obtained in the same way. Beside the usual bosonization rules and the usual Mandelstam fermionic representations, we also got the Hamiltonian ones, expressing the currents $J_\pm$ in terms of both the coordinate φ and momentum π. These rules are more general, because they are valid for arbitrary Hamiltonian and they are a consequence of the commutation relations. After elimination of the momenta on invoking their equations of motion, we came back to the conventional bosonization rules and to the conventional Mandelstam fermionic representations. The Schwinger term and consequently the sine-Gordon action have the correct dependence on Planck's constant $\hbar$, because κ is proportional to $\hbar$. The fact that $\hbar$ arises in the classical effective theory and in the coupling constant relation shows the quantum origin of the established equivalence.

A Normal ordering and central term

In this appendix we will derive commutation relations of the current operators $\hat{j}_\pm \equiv -\mathrm{i}\sqrt{2} :\hat{\pi}_\pm \hat{\psi}_\pm:$

$$\left[\hat{j}_\pm(\sigma), \hat{j}_\pm(\bar{\sigma})\right] = \pm 2\mathrm{i}\hbar\kappa\delta'(\sigma-\bar{\sigma}),$$
$$\left[\hat{j}_\pm(\sigma), \hat{j}_\mp(\bar{\sigma})\right] = 0\,, \tag{A.1}$$

where $\kappa \equiv \frac{\hbar}{2\pi}$.

The current operators $\hat{j}_\pm$ we define using the normal ordering prescription. In order to decompose these operators in positive and negative frequencies in position space, let us introduce the two parts of the delta function

$$\delta^{(\pm)}(\sigma) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\,\theta(\mp k)\,\mathrm{e}^{\mathrm{i}k(\sigma\mp\mathrm{i}\varepsilon)} = \frac{\mp\mathrm{i}}{2\pi(\sigma\mp\mathrm{i}\varepsilon)} \quad (\varepsilon>0)\,, \tag{A.2}$$

where θ is the unit step function. They obviously obey the relation $\delta(\sigma)=\delta^{(+)}(\sigma)+\delta^{(-)}(\sigma)$ and have the following properties:

$$\delta^{(+)}(\sigma) = \delta^{(-)}(-\sigma)\,, \tag{A.3}$$

$$\left[\delta^{(+)}(\sigma)\right]^2 - \left[\delta^{(-)}(\sigma)\right]^2 = \frac{\mathrm{i}}{2\pi}\,\delta'(\sigma)\,. \tag{A.4}$$

Any operator $\hat{O}$ can also be decomposed in two parts

$$\hat{O}^{(\pm)}(\tau,\sigma) = \int_{-\infty}^{\infty} \mathrm{d}\bar{\sigma}\,\delta^{(\pm)}(\sigma-\bar{\sigma})\,\hat{O}(\tau,\bar{\sigma})\,, \tag{A.5}$$

so that $\hat{O}=\hat{O}^{(+)}+\hat{O}^{(-)}$. We promote the operators $\hat{\pi}_+^{(-)}$ and $\hat{\psi}_+^{(-)}$ to annihilation ones, and the operators $\hat{\pi}_+^{(+)}$ and $\hat{\psi}_+^{(+)}$ to creation ones,

$$\hat{\pi}_+^{(-)}|\,0\rangle = \hat{\psi}_+^{(-)}|\,0\rangle = 0\,, \qquad 0|\,\hat{\pi}_+^{(+)} = \langle 0|\,\hat{\psi}_+^{(+)} = 0\ . \tag{A.6}$$

In order to preserve the symmetry under parity transformations, we define creation and annihilation operators for $\hat{\pi}_-$ and $\hat{\psi}_-$ in an opposite way [operators with index $(-)$ are creation and ones with index $(+)$ are annihilation operators]. The normal order for a product of the operators means that creation operators are placed to the left from the annihilation ones.

From the basic commutation relations

$$\left[\hat{\psi}_\pm(\sigma), \hat{\pi}_\pm(\bar{\sigma})\right] = \mathrm{i}\hbar\,\delta(\sigma-\bar{\sigma})\,, \tag{A.7}$$

we have

$$\left[\hat{\psi}_+^{(\pm)}(\sigma), \hat{\pi}_+^{(\mp)}(\bar{\sigma})\right] = \mathrm{i}\hbar\,\delta^{(\pm)}(\sigma-\bar{\sigma})\,, \tag{A.8}$$

$$\left[\hat{\psi}_-^{(\pm)}(\sigma), \hat{\pi}_-^{(\mp)}(\bar{\sigma})\right] = \mathrm{i}\hbar\,\delta^{(\pm)}(\sigma-\bar{\sigma})\,, \tag{A.9}$$

and the other commutation relations are trivial. Since the Poisson brackets for the currents $j_\pm$ vanish, the only possible difference between the classical and quantum algebra is the appearance of a central term at the quantum level. Because of this, we find the form of the current algebra taking the vacuum expectation value of the commutators,

$$\left[\hat{j}_\pm(\sigma), \hat{j}_\pm(\bar{\sigma})\right] = \varDelta_\pm(\sigma,\bar{\sigma}) - \varDelta_\pm(\bar{\sigma},\sigma)\,, \tag{A.10}$$

where $\varDelta_\pm(\sigma,\bar{\sigma}) \equiv \langle 0|\,\hat{j}_\pm(\sigma)\hat{j}_\pm(\bar{\sigma})|\,0\rangle$.

Using the fact that the operators $\hat{j}_\pm$ are normal ordered, the only non-trivial contributions have the form

$$\varDelta_\pm(\sigma,\bar{\sigma}) = -2\langle 0|\,\hat{\pi}_\pm^{(\mp)}(\sigma)\hat{\psi}_\pm^{(\mp)}(\sigma)\hat{\pi}_\pm^{(\pm)}(\bar{\sigma})\hat{\psi}_\pm^{(\pm)}(\bar{\sigma})|\,0\rangle = -2\hbar^2\left[\delta^{(\mp)}(\sigma-\bar{\sigma})\right]^2\,. \tag{A.11}$$

With the help of the $\delta^{(\pm)}$ function properties (A.4) we obtain (A.1). The commutator for the currents with different lower indices do not have a central term, and neither do commutators of the currents with the operators $\hat{t}_\pm$ and $\hat{\rho}_\pm$.

B Regularization of the field products

In this appendix, we will show, using a regularization procedure, that the following relations hold:

$$\sqrt{2}\hat{\Psi}^*_\pm(\sigma)\hat{\Psi}_\pm(\sigma+\eta)|_{\eta\to 0} \equiv \hat{\hat{J}}_\pm(\sigma,\sigma+\eta)|_{\eta\to 0} = \frac{F_\pm}{\eta \pm \mathrm{i}\varepsilon} + Z_\pm \hat{J}_\pm \quad (\varepsilon > 0), \tag{B.1}$$

where $F_\pm$ and $Z_\pm$ are given by (Λ is the cutoff parameter)

$$F_\pm = \pm \mathrm{i}\Lambda\sqrt{2}\,|C_\pm|^2\,, \qquad Z_\pm = \frac{\Lambda}{\kappa}\,|C_\pm|^2\,. \tag{B.2}$$

Starting with the definition of the operator $\hat{\hat{J}}_\pm$ and using the formula ((3.5) in [2])

$$:\mathrm{e}^{\hat{A}}::\mathrm{e}^{\hat{B}} = \mathrm{e}^{[\hat{A}^{(+)}\hat{B}^{(-)}]} :\mathrm{e}^{\hat{A}+\hat{B}}:\,, \tag{B.3}$$

($[A^{(+)}, B^{(-)}]$ is a c-number), we get

$$\hat{\hat{J}}_\pm(\sigma,\bar{\sigma}) = \sqrt{2}|C_\pm|^2 \exp\left\{\frac{1}{2\kappa^2}X_\pm(\sigma,\bar{\sigma})\right\} \times :\exp\left\{\pm\frac{\mathrm{i}}{\kappa\sqrt{2}}\left[\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})\right]\right\}:\,, \tag{B.4}$$

where

$$X_\pm(\sigma,\bar{\sigma}) \equiv \left[\hat{I}^{(\pm)}_\pm(\sigma), \hat{I}^{(\mp)}_\pm(\bar{\sigma})\right]\,. \tag{B.5}$$

We compute the commutators $X_\pm$ using the algebra of the operators $\hat{I}_\pm$

$$\left[\hat{I}_\pm(\sigma), \hat{I}_\pm(\bar{\sigma})\right] = \int_{-\infty}^{\sigma}\mathrm{d}\sigma_1 \int_{-\infty}^{\bar{\sigma}}\mathrm{d}\sigma_2 \left[\hat{J}_\pm(\sigma_1), \hat{J}_\pm(\sigma_2)\right]\,. \tag{B.6}$$

The last relation can be rewritten in the form

$$\left[\hat{I}_\pm(\sigma), \hat{I}_\pm(\bar{\sigma})\right] = \pm\mathrm{i}\hbar\kappa \int_{-\infty}^{\sigma}\mathrm{d}\sigma_1 \int_{-\infty}^{\bar{\sigma}}\mathrm{d}\sigma_2 \times\left[\partial_{\sigma_1}\delta(\sigma_1-\sigma_2) - \partial_{\sigma_2}\delta(\sigma_1-\sigma_2)\right]\,, \tag{B.7}$$

which is obtained from the algebra of the currents $\hat{J}_\pm$. Performing the integration we get

$$\left[\hat{I}_\pm(\sigma), \hat{I}_\pm(\bar{\sigma})\right] = \mp\mathrm{i}\hbar\kappa\varepsilon(\sigma-\bar{\sigma})\,, \tag{B.8}$$

where $\varepsilon(\sigma) = \theta(\sigma) - \theta(-\sigma)$ is the sign function. From the expression

$$\left[\hat{I}^{(\pm)}_\pm, \hat{I}_\pm\right] = \left[\hat{I}^{(\pm)}_\pm, \hat{I}^{(\mp)}_\pm\right] \tag{B.9}$$

it follows that

$$X_\pm(\sigma,\bar{\sigma}) \equiv \left[\hat{I}^{(\pm)}_\pm(\sigma), \hat{I}^{(\mp)}_\pm(\bar{\sigma})\right] = \mp\mathrm{i}\hbar\kappa\int_{-\infty}^{\infty}\mathrm{d}\sigma_1\,\delta^{(\pm)}(\sigma-\sigma_1)\,\varepsilon(\sigma_1-\bar{\sigma}). \tag{B.10}$$

Using (A.2), the last relation gets the form

$$X_\pm = \mp\mathrm{i}\hbar\kappa\int_{-\infty}^{\infty}\mathrm{d}\sigma_1\,\frac{\mp\mathrm{i}}{2\pi(\sigma-\sigma_1\mp\mathrm{i}\varepsilon)}\,\varepsilon(\sigma_1-\bar{\sigma})\,. \tag{B.11}$$

After regularization (Λ is the cutoff parameter) the integral on the right hand side takes the form

$$X_\pm = -\kappa^2\left\{\int_{\bar{\sigma}}^{\Lambda}\mathrm{d}\sigma_1\,\frac{1}{\sigma-\sigma_1\mp\mathrm{i}\varepsilon} - \int_{-\Lambda}^{\bar{\sigma}}\mathrm{d}\sigma_1\,\frac{1}{\sigma-\sigma_1\mp\mathrm{i}\varepsilon}\right\}\,. \tag{B.12}$$

Computing the integral and taking $\Lambda \gg \sigma$, we get

$$X_\pm = \kappa^2\left\{\ln\left[\frac{\Lambda^2}{(\bar{\sigma}-\sigma\pm\mathrm{i}\varepsilon)^2}\right] \pm \mathrm{i}\pi\right\}\,. \tag{B.13}$$

The last equation implies

$$\mathrm{e}^{\frac{1}{2\kappa^2}X_\pm(\sigma,\bar{\sigma})} = \frac{\mp\mathrm{i}\Lambda}{\sigma-\bar{\sigma}\mp\mathrm{i}\varepsilon} = 2\pi\Lambda\delta^{(\pm)}(\sigma-\bar{\sigma})\,, \tag{B.14}$$

$$\mathrm{e}^{-\frac{1}{2\kappa^2}X_\pm(\sigma,\bar{\sigma})} = \pm\mathrm{i}\,\frac{\sigma-\bar{\sigma}\mp\mathrm{i}\varepsilon}{\Lambda}\,. \tag{B.15}$$

Substituting (B.14) in (B.4), we get the regularized form of the operator $\hat{\hat{J}}_\pm$

$$\hat{\hat{J}}_\pm(\sigma,\bar{\sigma}) = \sqrt{2}|C_\pm|^2\frac{\mp\mathrm{i}\Lambda}{\sigma-\bar{\sigma}\mp\mathrm{i}\varepsilon} \times :\exp\left\{\pm\frac{\mathrm{i}}{\kappa\sqrt{2}}\left[\hat{I}_\pm(\sigma)-\hat{I}_\pm(\bar{\sigma})\right]\right\}: \quad (\varepsilon > 0). \tag{B.16}$$

For the infinitesimal $\eta = \bar{\sigma} - \sigma$ we get

$$\hat{\hat{J}}_\pm(\sigma,\sigma+\eta)|_{\eta\to 0} = \sqrt{2}\,|C_\pm|^2\,\frac{\pm\mathrm{i}\,\Lambda}{\eta\pm\mathrm{i}\varepsilon} + \frac{|C_\pm|^2\,\Lambda}{\kappa}\,\hat{J}_\pm \quad (\varepsilon > 0)\,, \tag{B.17}$$

which is exactly the relation (B.1), and $F_\pm$ and $C_\pm$ are given by (B.2).

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